Progressive Collapse of Rigid-Plastic Circular Foundations

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Abstract: The paper presents an analytical study of the behavior of a rigid-perfectly plastic circular foundation plate indenting an elastic two-parameter soil layer (Vlasov soil) under the action of a statically increasing applied load. Closed-form solutions are obtained for critical loads, maximum surface settlements, foundation deflections and soil reactions in terms of the load intensity, after the plate has been transformed into a mechanism. Two distinct phases of metaplastic behavior are identified depending on whether or not full contact is maintained between soil and foundation. The results are compared with those obtained for a Winkler space and an elastic continuum halfspace.

Introduction

The interaction between foundations and supporting soil media has been a subject of keen interest to both geotechnical and structural engineering for a long time. In recent years, solutions to various problems of beams, plates, and shells continuously supported by deformable media have also been in great demand in other branches of engineering, as, e.g., aerospace and mechanical engineering (10,17). Since a complete analysis of such an interaction problem is a formidable task, the interest of engineers has usually been restricted to predicting stresses and displacements in the structural foundation and contact stresses at the soil-foundation interface.

A variety of theoretical formulations can be employed in the investigation of such interaction phenomena. The usual approach is based on the inclusion of the soil reactions into the corresponding differential equation of the beam, plate, or shell. Since these reactions depend on the complicated behavior of the soil medium, their determination constitutes the primary difficulty of the problem. In practice, when designing relatively unimportant structures, this difficulty is overcome by adopting an arbitrary simplification, e.g., assuming that the contact pressure is linearly distributed over the soil-foundation interface. This assumption in essence ignores the existence of the problem since it does not account for the compatibility of deformations between soil and structures; it will not be further addressed in this paper.

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The simplest representation of a continuous soil medium has been proposed by Winkler (21) who described the soil as a system of continuously distributed independent linear springs which offer resistance in the direction of their axis only. A rigid foundation carrying a load on such a medium will be resisted by uniform contact stresses, while the settlement of the soil surface outside the loaded region remains equal to zero. On the other hand, the soil was considered as a semi-infinite linearly elastic continuum, a mathematically much more difficult problem. A number of solutions are available in the literature for an isotropic or an anisotropic continuum (6,7,16,17). Although such a representation constitutes an improvement over the Winkler model, in that it allows estimation of the spatial distribution of stresses and displacements within the soil, it is by no means an exact description of reality; e.g., it predicts surface displacements away from the loaded area that decrease much slower than actual observations show.

To bridge the gap between these two extreme cases, a number of soil models has been developed either by asuming some kind of interaction between the springs of the Winkler base or by introducing simplifying assumptions with respect to expected displacements and stresses in the elastic continuum; Kerr (10) and Selvadurai (17) have presented comprehensive critical reviews of various proposed soil models.

An interesting model has been advocated by Vlasov (18) who approached the problem from a "continuum" point of view, imposing certain reasonable restrictions upon the possible deformations of an elastic layer and using a variational method to derive governing differential equations of a variety of soil-structure interaction problems. In its simplest (and most popular) form, Vlasov's model involves two parameters. In many cases, it is believed (17) that it can simulate certain aspects of the behavior of soil-foundation systems more realistically than the other two soil models (Winkler base and elastic continuum) and, above all, it leads to significantly simpler solutions than those of the theory of elasticity. This explains the wide application of the model, as evidenced by studies published in this country (8,20,22). The scope of the present paper is to employ the two-parameter version of Vlasov's model to study the post-plastic behavior of circular plate foundations supported by a single soil-stratum on rigid rock.

In order to better utilize the material strength and to meet the growing needs of increasing loads, it is universally accepted that plastic or ultimate strength methods of design be used in preference to elastic methods. Thus, application of the plastic theory of structures in the design of foundations has been a subject of engineering interest over the last years (1,4,5,12,13,14,15,23,24,25). It has been shown (4) that plastic theories not only lead to simpler methods of analysis, but, moreover, they give results which are more reliable and less sensitive to the exact contact stress distribution than those of the elastic theories.

Zingone, in a series of publications (23,24,25) investigated the collapse load of rigid-plastic foundation plates on a plastic Winkler-type soil; i.e., he assumed that both structure and soil are in a state of plastic equilibrium. Along the same lines, Meyerhoff and Rao (14) empirically combined the theoretical collapse load of a rigid-plastic footing on elastic Winkler base with the soil-failure load from bearing capacity theory to estimate the collapse load of a plastic foundation on plastic soil. Since, however, soil is a more unreliable material than either concrete or steel, it should be assigned a higher safety factor. Thus, at the time of collapse of the structural component of a footing, soil pressure will, in general, be much lower than the ultimate bearing capacity of soil. This philosophy underlines the published studies of foundation collapse on elastic supporting soil (1,3,4,12,13,15).

The present paper describes an analytical method to study the behavior of a rigid-perfectly plastic circular foundation plate indenting a two-parameter foundation soil under the action of a slowly incrementing externally applied load. Closed-form solutions are obtained for the maximum settlement, the shape of the deformed surface, and the soil reactions, in terms of the applied load, after the foundation slab has been transformed into a mechanism. Two distinct phases of "meta-plastic" foundation behavior are identified, depending on whether or not full contact is maintained between structure and soil. It is shown that beyond a critical load the foundation lifts off the ground and strong geometric nonlinearities are observed. "Transversality" conditions for determining the size of the contact area are developed, and the rate of slab deflection is graphically demonstrated. The results are compared with those obtained for a Winkler base (4) or an elastic continuum (12).

Two-Parameter Soil Model

A uniform soil layer of thickness H underlain by rigid rock and subjected to vertical axisymmetric loading at the surface is considered (Fig. 1(a)). If the vertical and horizontal displacements, $w(r,z)$ and $u(r,z)$, can be found at all points in the layer, stresses and strains will be obtained from established stress-strain and strain-deformation relations of the theory of elasticity. In problems, such as the analysis of soil-foundation interaction, horizontal displacements, $u(r,z)$, may be considered of negligible magnitude in comparison with vertical displacements. Then, in order to obtain an approproximate solution, the unknow function $w(r,z)$ is expanded in finite series:

$$w(r,z) = \sum_{i=1}^{m} W_i(r)\, h_i(z) \quad \text{. . . . . (1)}$$

in which the dimensionless functions, $h_i(z)$, assumed to be known, represent the distribution of vertical displacements with depth from the surface, while the unknown functions, $W_i(r)$, represent, in essence, the settlement of the surface. For a single uniform layer, Eq. 1 can be further simplified to

$$w(r,z) = W(r)\, h(z) \quad \text{. . . . . (2)}$$

since a single function, $h(z)$, can be found to describe with sufficient accuracy the variation of vertical displacements and normal stresses with depth; e.g., in a relatively shallow layer loaded by a large circular foundation, e.g., $R/H \geq 1$, vertical normal stresses and strains are nearly constant, and, therefore, the displacements w decrease linearly with depth. Thus a function

$$h(z) = 1 - \frac{z}{H} \quad \text{. . . . . (3)}$$

will adequately describe the true variations of w, ϵ_z, and σ_z in the z direction.

In a deeper layer, e.g., $R/H < 1/2$, the variation of stresses, strains, and displacements can be described by a function

$$h(z) = \frac{\sinh \lambda \dfrac{H - z}{R}}{\sinh \lambda \dfrac{H}{R}} \qquad (4)$$

in which the parameter λ determines the rate of decrease of displacements with depth and can be selected for each particular problem on the basis of experimental or published theoretical data of normal stress distributions (16).

Using Langrange's principle of virtual displacements and the elastic stress-strain-deformation relations with the displacement of all points expressed by Eq. 2, the equilibrium equation of the soil layer subjected to an axisymmetrically distribution load, $q(r)$, on the surface (Fig. 1) is obtained (p. 39 of Ref. 18):

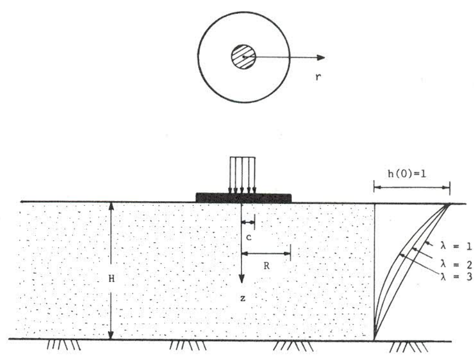

FIG. 1.—Single-Layer Two-Parameter Soil Model and Vertical Displacement Distribution Function

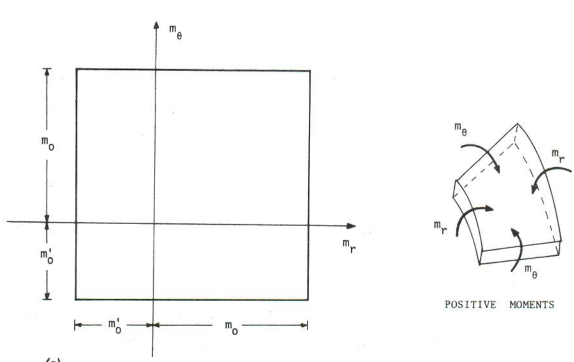

FIG. 2.—Yield Criterion of Foundation Plate

$$-2t\,\nabla^2 W(r) + k\,W(r) = q(r)$$

in which $$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr} \quad (5)$$

is the Laplace operator in cylindrical coordinates, and

$$k = D\int_0^H \left[\frac{dh(z)}{dz}\right]^2 dz; \quad t = \frac{G}{2}\int_0^H h(z)^2 dz \quad (6)$$

are the two elastic parameters of the foundation, with D and G = the constrained and shear modulus of soil, respectively. It is seen that the two parameters, hereafter referred to as dilatational and shear parameters, depend on the selected vertical displacement function, $h(z)$. Appendix I shows the expressions k and t corresponding to displacement functions described by Eqs. 3 and 4.

Foundation, Loading, and Phases of Deformation

A thin rigid-perfectly plastic circular foundation plate rests on the surface of a two-parameter soil layer. The interface between foundation and soil is assumed to be frictionless (''smooth'' or ''relaxed'' boundary), and the plate material obeys the square yield criterion of Fig. 2 with the associated flow rule.

A uniformly distributed load is slowly applied over a small circle of radius, C, concentric with the foundation plate so that axial symmetry is preserved. Thereafter, the load is gradually incremented, and the plate is driven into the soil layer as a rigid flat indenter until fully-plastic radial moments are realized, and a plastic mechanism develops in the plate. With further increase of the load, the plate deforms into a conical surface, but no sudden failure is observed; i.e., although the external forces continue to increase, equilibrium is maintained be-

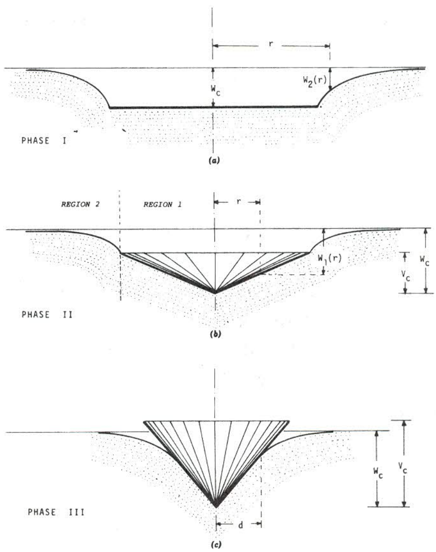

FIG. 3.—Three Phases of Deformation of Rigid-Plastic Circular Foundation

cause soil reactions also increase. At a certain level of the load intensity, a change in the geometry of the soil-foundation interface takes place, as the foundation lifts off the base. Thereafter, the rate of settlement under the applied external load grows rapidly until, eventually, the foundation "sinks" into the ground.

The interest of the paper is to study this "progressive collapse" by developing analytical expressions relating foundation settlement and soil reactions to any applied total external force. For reasons of clarity of the presentation, three distinct phases of foundation deformation are identified, as portrayed in Fig. 3:

1. For small values of the external load, $P < P_o$, the bending moments that develop in the foundation plate are smaller than the corresponding plastic moments, m_o or m_o'. Therefore, no yielding takes place, and the foundation penetrates the soil layer as a rigid punch (Phase I).
2. Beyond a critical threshold value of the load, P_o, [named after Augusti (1)], an infinite number of radial yield lines develop, and the plate deforms into a conical shape. Full contact is maintained between foundation and soil during this phase (Phase II).
3. Beyond a certain value of the load, P_s, named separation load by Krajcinovic (12), the foundation lifts off the soil as the center of the plate is driven further into the ground, while the edges move upward (Phase III).

Governing Equations and Solution

Constant Contact Area (Phase II).—Two different regions can be distinguished in this case, one under and one beyond the foundation. Let W_1 and W_2 = the (unknown) settlements in the two regions (measured from the original ground surface), and W_c = the settlement of the foundation center. By Eq. 5, the two differential equations governing the spatial variation of W_1 and W_2 are

$$\text{at} \quad 0 \le r \le R: \quad \frac{d^2W_1}{dr^2} + \frac{1}{r}\frac{dW_1}{dr} - \alpha^2 W_1 = -\frac{\sigma_o(r)}{2t} \qquad (7)$$

$$\text{at} \quad R \le r < \infty: \quad \frac{d^2W_2}{dr^2} + \frac{1}{r}\frac{dW_2}{dr} - \alpha^2 W_2 = 0 \qquad (8)$$

in which $\sigma_o(r)$ = the unknown as yet contact stress distribution at the foundation-soil interface; and $\alpha^2 = k/2t$. Eqs. 7 and 8 express in an integral form the equilibrium conditions for the soil. The governing equation for the foundation structure can be conveniently derived using the principle of virtual work. Allowing the slab mechanism to deflect a virtual displacement such that the center moves an arbitrary distance, e.g., unity, the internal and external works are estimated and equated. This leads to

$$2\pi m_o = P\left(1 - \frac{2}{3}\beta\right) - 2\pi \int_0^R \sigma_o(r)\, r\left(1 - \frac{r}{R}\right)dr \qquad (9)$$

in which $\beta = c/R$ and $P = p\pi c^2$ = total applied load.

Eqs. 7–9 constitute a system of three differential-integral equations with three unknowns (W_1, W_2, and σ_o). Eq. 8 can be directly solved for W_2:

$$W_2(r) = CK_o(\alpha r) + C'I_o(\alpha r) \quad \text{(10a)}$$

in which K_o and I_o = the modified Bessel functions of order zero, second and first kind, respectively [see Watson (19)]. Since W_2 must vanish at infinity, $C' = 0$; $I_o \rightarrow \infty$ as $r \rightarrow \infty$. Thus

$$W_2 = CK_o(\alpha r) \quad \text{(10b)}$$

To integrate Eq. 7 we observe that, due to the rigid-plastic behavior of the plate, W_1 can be expressed in terms of W_c and the slope θ of the deformed foundation:

$$W_1 = W_c - \theta r \quad \text{(11)}$$

Introducing Eq. 11 in Eq. 7 yields the form of the contact stress distribution:

$$\sigma_o(r) = kW_c + \frac{2t\theta}{r} - k\theta r \quad \text{(12)}$$

which, upon substitution in Eq. 9, results in

$$\left(1 - \frac{2\beta}{3}\right)P = 2\pi m_o + \frac{\pi k W_c R^2}{3} + \pi R\left(2t - \frac{kR^2}{6}\right)\theta \quad \text{(13)}$$

which, for a given plastic moment of the foundation, m_o, relates the applied load, P, to the resulting deformations W_c and θ.

In addition to the distributed soil reactions against the foundation that are described by Eq. 12, fictitious reactions, Q_f, per unit length act along the contour of the circular plate. These are due to the deformations of the soil beyond the plate region, and correspond to the infinitely large stresses beneath the edges of rigid foundations predicted by the theory of elasticity for a semi-infinite continuum (Ref. 16, p. 166). To see how they are "created," consider an infinitesimally thin hollow cylinder of soil with height, H; internal radius, $R - f$; and external radius, $R + f$ in which $f \rightarrow 0$. The condition of equilibrium can be written by equating to zero the total work done by all forces acting on the cylinder for a virtual displacement, $\delta w(r,z) = h(z)$:

$$2\pi R \cdot Q_f \cdot h(o) + \int_0^H \int_0^{2\pi} h(z) \cdot \tau_{rz}^{(2)} \cdot (R + f) d\phi dz - \int_0^H \int_0^{2\pi} h(z) \cdot \tau_{rz}^{(1)}$$
$$\cdot (R - f) d\phi dz = 0 \quad \text{(14)}$$

in which the shear stresses $\tau_{rz}^{(1)}$ and $\tau_{rz}^{(2)}$, corresponding to the two regions of Fig. 3(*b*), are obtained from the elastic relation $\tau_{rz} = G(\partial w/\partial r)$ and Eq. 2. Substitution in Eq. 14 and integration yields the fictitious force:

$$Q_f = 2t\left(\frac{dW_1}{dr} - \frac{dW_2}{dr}\right)_{r=R} \quad \text{(15)}$$

Thus, Q_f is caused by the different slopes of the settling surface at the edge of the foundation, and the ability of the soil to take up shearing stresses.

The boundary conditions of the problems can now be stated as follows:

$$W_1(R) = W_2(R) \quad \text{(16)}$$

$$P = 2\pi \int_0^R \sigma_o(r)\, r\, dr + 2\pi R Q_f \quad \text{(17)}$$

and the system of six equations (Eqs. 10*b*, 12, 13, 15, 16, and 17) can be analytically solved for the six unknown quantities, W_2, C, W_c, θ, Q_f, and σ_o.

After some lengthy but straightforward algebraic operations, the following closed-form relations are derived:

$$W_c = \left(\frac{P\xi}{k\pi R^2} - \frac{2m_o}{kR^2}\right)\Omega \quad \text{(18a)}$$

$$\text{with} \quad \xi = 1 - \frac{2}{3}\beta + \frac{3 - \dfrac{(\alpha R)^2}{2}}{g(\alpha R)^2} \quad \text{(18b)}$$

$$\Omega = \frac{g}{\dfrac{(g-1)}{6} + \dfrac{(g+1)}{(\alpha R)^2}} \quad \text{(18c)}$$

$$\text{and} \quad g = 2\left[1 + 3\frac{K_1(\alpha R)}{\alpha R K_o(\alpha R)}\right]; \quad \text{(18d)}$$

$$\theta = \frac{1+g}{g}\frac{W_c}{R} - \frac{3}{g}\frac{P}{k\pi R^3} \quad \text{(19)}$$

$$W_2 = \frac{W_c - \theta R}{K_o(\alpha R)} K_o(\alpha r); \quad R < r < \infty \quad \text{(20)}$$

Eqs. 18–20 can be used to compute the deformation of the soil surface for any applied load, P, given the moment capacity of the plate, m_o, and provided

$$P_o \le P \le P_s \quad \text{(21)}$$

The contact stress distribution $\sigma_o(r)$ is then obtained from Eq. 12 after substituting the computed values of W_c and θ.

Threshold Load.—The load, P_o, required to transform the foundation plate into a mechanism and, thus, bring it into the second phase of deformation is obtained by setting $\theta = 0$ in the previous relations. Calling W_o the threshold settlement of the plate at this particular load, $W_o = W_1 = W_c$, Eqs. 18 and 19 yield

$$P_o = \frac{3\pi}{1.5 - \beta - \dfrac{0.5}{1 + 2\dfrac{K_1(\alpha R)}{\alpha R K_o(\alpha R)}}} m_o \quad \text{(22)}$$

for the threshold load, and

$$\tilde{W}_o \equiv \frac{W_o}{\dfrac{P_o}{(k\pi R^2)}} = \frac{1}{1 + 2\dfrac{K_1(\alpha R)}{\alpha R K_o(\alpha R)}} \qquad (23)$$

for the threshold settlement. It is quite interesting to compare the above expressions for P_o and W_o with those resulting from the alternate soil models described in the introduction, namely the Winkler model and the continuum model. Gazetas and Tassios (4) have obtained P_o of a plate on Winkler soil:

$$P_{o\text{Winkler}} = \frac{3\pi}{1 - \beta} m_o \qquad (24)$$

while Krajcinovic (12) reported for a semi-infinite continuum

$$P_{o\text{continuum}} \simeq \frac{3\pi}{1.18 - \beta} m_o \qquad (25)$$

Fig. 4 portrays the reduced threshold load, P_o/m_o, predicted from Eqs. 22, 24, and 25, as a function of the reduced radius of loading, β. Since α in Eq. 22 is a function of the relative thickness, H/R, of the soil layer, the empirical parameter, λ, and Poisson's ratio, ν, of the soil, a family of curves is displayed in this figure for the presented soil model. H/R ranges from 1–4, λ from 1–2 while $\nu = 0.30$. Only a single curve is obtained for the Winkler soil or the half-space continuum. The agreement of the three models ranges, in general, from satisfactory at low β ratios, i.e., for nearly concentrated load, to rather poor at very

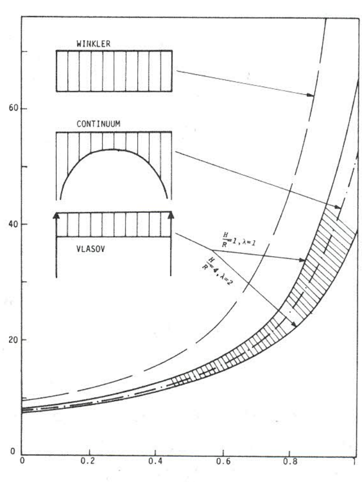

FIG. 4.—Comparison of Reduced Threshold Loads and Corresponding Contact Stress Distributions from Three Soil Models

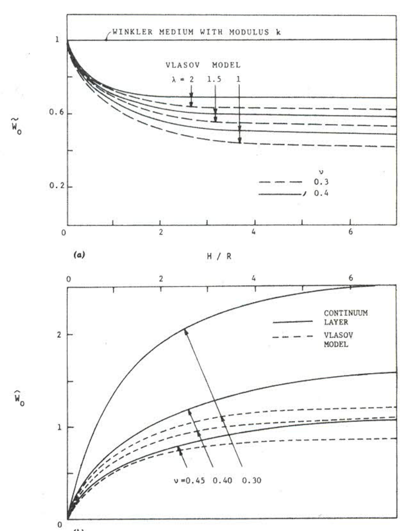

FIG. 5.—Comparison of Dimensionless Maximum Settlements from Three Soil Models

high β ratios, i.e., for uniform applied pressure over almost the whole foundation area. Notice, however, that the presented (Vlasov, Ref. 18) model leads to threshold loads that are much closer to those predicted for a continuum throughout the range of β. In fact, Eq. 25 plots almost in the center of the band comprising the curves of Eq. 22.

These similarities and discrepancies among the three models can be qualitatively explained if one considers the three corresponding contact stress distributions, shown also in Fig. 4. It is obvious that the concentrated forces, Q_f, at the contour of a plate pushed into a Vlasov soil and the infinite stresses at the edges of a rigid circular slab supported by an elastic homogeneous half space produce higher bending moments and, thus, transform the foundation into a mechanism faster, i.e., at a smaller load, than the uniform reactions of a Winkler soil.

Fig. 5(*a*) displays W_o as a function of the reduced thickness H/R. The various curves were computed from Eq. 23 for $\nu = 0.3$, 0.4 and three different values of λ (1, 1.5, and 2). Also shown for comparison is the normalized threshold settlement of a Winkler soil characterized by a constant subgrade modulus taken equal to k. The discrepancy of the two theories is small for very shallow layers, but rapidly increases to approx 30–60% (depending on the choice of λ), as soon as H/R exceeds 1.5–2.0. Notice, though, that for the Vlasov model, k is not constant but varies with H/R (Appendix I).

To compare with the threshold settlements of an elastic continuum, it is convenient to define

$$\hat{W}_o = \frac{W_o}{P_o} \pi R D \qquad (26)$$

in which $D = E(1 - \nu)/[(1 + \nu)(1 - 2\nu)]$ is the constrained soil modulus. For the Vlasov model (Eq. 23), $\hat{W}_o$ increases with H/R and ν, as shown in Fig. 5(*b*). For an elastic finite continuum having the same E and ν, exact analytical solution for W_o does not exist. Recently, Kausel and Ushijima (9), on the basis of extensive finite-element analyses, suggested that W_o can be obtained with very good accuracy from

$$W_o \simeq \frac{P_o(1 - \nu^2)}{2ER}\left(1 + 1.28\frac{R}{H}\right)^{-1} \qquad (27)$$

$\hat{W}_o$ obtained from Eq. 27 is also plotted in Fig. 5(*b*) for three values of Poisson's ratio, $\nu = 0.3$, 0.4, and 0.45. The discrepancy of the two models increases with layer thickness and Poisson's ratio. In fact, the two-parameter model predicts $\hat{W}_o$, which only slightly changes with ν, whereas for the elastic continuum, ν is a critical parameter. The phenomenon is quite understandable, in view of the significant lateral displacements that develop according to the theory of elasticity in the soil as $\nu \rightarrow 0.5$, especially for large H/R ratios. Such lateral displacements are not allowed in the two-parameter model which, consequently, behaves as a stiffer medium.

Lifting-Off Phase.—Since no tensile stresses can develop between foundation and soil, beyond a critical load, P_s, the foundation lifts off the ground as is shown in Fig. 3(b). Problems of similar nature, involving boundaries with variable geometry, lead to nonlinear force-deformation relations. Studies accounting for the lift-off of beams or plates supported on a continuous base have been published for both elastic (2,11) and plastic (4,12,23,24,25) structures. A complete solution must yield not only foundation settlements and soil reactions but also "transversality" conditions for locating the position of the variable contact surface. In our case, due to axial symmetry, the latter requirement is translated as "determination of radius d" (Fig. 3(c)).

Eqs. 7, 8, and 11 are obviously still holding true, while Eq. 9 changes to

$$2\pi m_o = P\left(1 - \frac{2}{3}\beta\right) - 2\pi \int_0^d \sigma_o(r)\, r\left(1 - \frac{r}{R}\right) dr \quad \ldots\ldots (28)$$

and Eq. 16 to $W_1(d) = W_2(d)$ .... (29)

in which W_2 = soil displacements beyond the contact area. Because of the nature of the two-parameter soil model, however, there seems to be some difficulty in formulating the additional needed boundary condition for the variable "free" edge of the foundation. In fact, Chernigovskaya (2) committed a serious error. Chernigovskaya studied the lifting off elastic foundation elements supported on "Pasternak" soil, whose governing differential equation is of the same form as Eq. 5 of the "Vlasov" model (10,11,16). There (2), it was stated that the distributed reaction stress, σ_o, must vanish at $r = d$. Although this seems to be intuitively true (and it is true for a Winkler base or an elastic continuum), it is not so with this soil model (as well as with that of Ref. 2). It can be easily seen from Eqs. 2 and 3 or 4, for example, that the model predicts nonzero normal vertical stresses even at points on the foundaion surface which carry no load. This is a result of employing the variational method, which applies the equilibrium conditions in integral form without providing for their fulfilment at every single point of the system. Instead, the correct boundary condition at $r = d$ is

$$Q_f = 0 \quad \ldots\ldots (30)$$

which is equivalent to stating that the slope of the deformed surface is continuous at the separation point of plate and soil. Note that Kerr (11) utilized a variational principle to rationally formulate the differential equation and mathematically well-posed boundary conditions (including the "transversality" condition) of a beam lifting off a "Pasternak" base. He also found that "at the separation point of beam and . . . layer there is no concentrated reaction force" (11).

Finally, Eq. 17 changes to

$$P = 2\pi \int_0^d \sigma_o(r) r dr \quad \ldots\ldots (31)$$

and the new set of equations can be solved for W_c, θ, and σ_o, in terms of the "contact" radius, d. The "transversality" condition (Eq. 39) is then used to determine d for each particular intensity of the applied force. After some lengthy but straightforward algebraic operations one arrives at

$$W_c = \frac{P}{\pi k R^2}\left(\frac{R}{d}\right)^2 (1 + ab)^{-1} \quad \ldots\ldots (32a)$$

$$\text{with} \quad a = \frac{2\alpha d \dfrac{K_1(\alpha d)}{K_o(\alpha d)}}{1 + \alpha d \dfrac{K_1(\alpha d)}{K_o(\alpha d)}} \quad \text{and} \quad b = \frac{1}{(\alpha d)^2} - \frac{1}{3} \quad \ldots\ldots (32b)$$

$$\theta = \frac{a}{2d} W_c \quad \ldots\ldots (33)$$

and $\sigma_o(r)$ is given by Eq. 12. The "transversality" condition is

$$\frac{P}{m_o} = \frac{12\pi}{6 - 4\beta - F(\alpha, d)} \quad \ldots\ldots (34)$$

$$\text{in which} \quad F(\alpha, d) = \left(6 - 2a + 6a(\alpha d)^{-2} - [4 - 1.5a + 3a(\alpha d)^{-2}]\frac{d}{R}\right) \cdot (1 + ab)^{-1} \quad \ldots\ldots (35)$$

For any applied load, $P \geq P_s$, and knowing the plastic moment of the plate, m_o, Eq. 34 yields by trial and error the "contact" radius, d. Eqs. 32, 33 and 12 will then fully specify the surface settlement, the foundation distortion, and the soil reactions.

To determine the "separation" load, P_s, it is sufficient to set $d = R$ in Eqs. 32, and 34. It is interesting to compare this load with the corresponding separation loads of a foundation on a Winkler base (4) or an elastic continuum (12)

$$P_{s\text{Winkler}} = \frac{12\pi m_o}{3 - 4\beta}; \quad P_{s\text{continuum}} = \frac{12\pi m_o}{\pi - 4\beta} \quad \ldots\ldots (36)$$

Fig. 6 compares the three reduced separation loads, P_s/m_o, portrayed as functions of β. The agreement between all of them is excellent, especially in the low β range (nearly concentrated loads). This is hardly surprising in view of the similar contact stress distributions predicted by the theories and shown also in Fig. 6. Note also that, as it can be confirmed with Eqs. 36 (or Fig. 6), no separation can occur if

$$\beta \gtrsim 0.75 - 0.80$$

i.e., with such a large loading area, the slab will never loose contact with the soil. On the other hand, for a concentrated load, i.e., $\beta = 0$, the separation load is only about 50% larger than the threshold load, irrespective of soil model.

The same excellent agreement is also observed with regard to the geometry of the contact area during the lifting-off phase, as predicted by the three mentioned theories. In essence, the reduced radius of contact, $\gamma = d/R$, is a unique function of P/P_s, irrespectively of soil model as Fig. 7(a) demonstrates.

However, some discrepancies exist among the surface settlements and slab deflections of the three theories. Fig. 7(b) displays in dimensionless form the evolution of the maximum surface settlement, $\hat{W}_c$, and the slab deflection, $\hat{V}_c$. The latter is obtained from

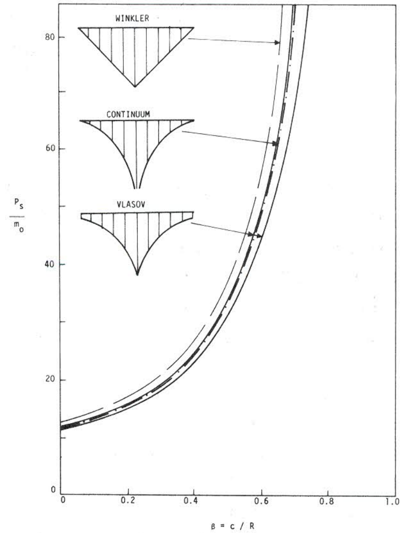

FIG. 6.—Comparison of Reduced Separation Loads and Corresponding Contact Stress Distributions from Three Soil Models

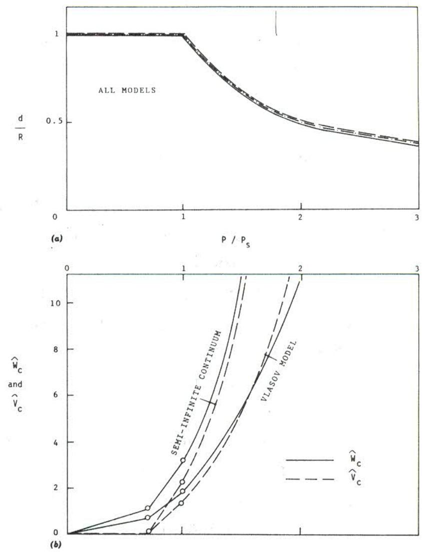

FIG. 7.—Evolution with Applied Total Load Of: (*a*) Soil-Foundation Contact Area; (*b*) Maximum Surface Settlement and Foundation Deflection

$$\hat{V}_c = \theta R \frac{\pi R D}{P_o} \quad \text{(37)}$$

in which θ is given by Eq. 33, and D = the constrained soil modulus. It is seen that below the threshold load $V_c = 0$, since up to that point the slab behaves as a rigid body. Thereafter, V_c grows at a much faster rate than W_c, as the foundation edges move upward under the influence of the soil reactions. Eventually, when $P \geq 1.22P_c$, V_c exceeds W_c while the radius of the contact circle, d, decreases below $0.82R$. Although the deformational behavior of the "collapsing" foundation on a continuum half space (12) is qualitatively similar with the one described previously, the predicted rate of increase of both W_c and V_c is a little faster (Fig. 7(*b*)). Note also that the Winkler model (4) leads to even larger values of these deformation quantities (not plotted in Fig. 7(*b*)).

One can easily explain why the present theory, which assumes "Vlasov" soil, slightly underestimates the settlements "observed" on an elastic continuum half-space: a cone penetrates the half space by laterally displacing the surface soil, rather than by compressing it. "Vlasov" soil, however, is effectively very stiff in the lateral direction and the only displacements that are "observed" are due to one-dimensional volumetric compression. The importance of such displacements diminishes as the contact area decreases and the cone becomes sharper.

Actual soils exhibit deformational anisotropy, being usually stiffer in the horizontal than the vertical direction (5,6). Consequently, one should expect the meta-plastic behavior of a foundation on actual soil to be something in between the predictions based on these two idealizations. Moreover, local yielding of soil is another important phenomenon at such extreme cases of deformation, and its proper evaluation is necessary for a more accurate prediction of foundation response.

Summary and Conclusions

The paper has presented an analytical study of the metaplastic response of an initially rigid circular foundation pushed into an elastic soil layer by a statically incrementing applied force. The soil is modeled as a two-parameter medium (Vlasov model) and closed-form expressions of pertinent response quantities are obtained by solving the governing differential equations while satisfying the well-posed boundary conditions. Two distinct phases of deformation are indentified. Beyond a critical, threshold load, P_o, the slab transforms into a mechanism through an infinite number of radial yield lines and deforms into a conical surface. No sudden failure is observed, as the load increases further, thanks to the also increasing soil resistance. When another critical load, $P_s \simeq 1.50\ P_o$, is exceeded, the distortion of the foundation plate becomes such that its edges lift off the ground. The subsequent settlement increases faster than the load (geometric nonlinearity), and it is felt that a prudent design should not allow such deformations to take place.

Expressions for the critical loads, maximum surface settlements, foundation deflections, and soil reactions are given as functions of load intensity for all phases of deformation. Extensive comparisons are made with results of two similar studies of collapsing foundations on elastic soil, modeled as a Winkler base (4) or as a continuum half space (12). It is concluded that good to excellent agreement exists with respect to the loads predicted by the three theories, especially the one presented here and that of Ref. 12 (continuum half space). However, some discrepancies are observed in the predictions of settlements and deflections, especially at the lifting-off phase. They are attributed to the large lateral deformations of the continuum which are expected to increase as Poisson's ratio approaches the limiting value of 1/2.

The method presented is believed to be useful in analyzing or designing mat foundations, and its extension to other foundation geometries seems feasible.

Appendix I.—Soil Parameters

For the displacement distribution function, $h(z)$, given by Eq. 3 the two elastic parameters are (18):

$$k = \frac{D}{H} = \frac{E(1-\nu)}{H(1+\nu)(1-2\nu)}; \quad t = \frac{GH}{6} = \frac{EH}{12(1+\nu)} \qquad (38)$$

and for $h(z)$ given by Eq. 4:

$$k = \frac{D\lambda}{2R}\,\frac{\sinh\left(\frac{\lambda H}{R}\right)\cosh\left(\frac{\lambda H}{R}\right) + \frac{\lambda H}{R}}{\sinh^2\left(\frac{\lambda H}{R}\right)};$$

$$t = \frac{GR}{4\lambda} \frac{\sinh\left(\frac{\lambda H}{R}\right)\cosh\left(\frac{\lambda H}{R}\right) - \frac{\lambda H}{R}}{\sinh^2\left(\frac{\lambda H}{R}\right)} \qquad (39)$$

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